Bäcklund Transformation Groups of Non-Linear Evolution Equations and the Painlevé Property

W.-H. Steeb
Theoretische Physik, Universität Paderborn
W. Oevel
Fachbereich Mathematik, Universität Paderborn

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We show that the group theoretical reduction of evolution equations which admit Lie-Bäcklund transformation groups does not lead in general to ordinary differential equations with the Painlevé property.

Recently several authors [1–7] have investigated the connection between non-linear evolution equations and the Painlevé property. The following conjecture has been stated: "Every non-linear ordinary differential equation resulting from a group theoretical reduction of a non-linear partial differential equation which can be solved by the inverse scattering method has the Painlevé property." Under the Painlevé property of an ordinary differential equation we understand the following: The only movable critical points of all its solutions are poles. We notice that a solution of an ordinary differential equation can have poles, essential singularities and branch points. Consequently, for an ordinary differential equation to have the Painlevé property we must require that there are no movable essential singularities or movable branch points.

Evolution equations which can be solved by the inverse scattering method are usually called soliton equations. Soliton equations have several remarkable properties in common: (I) the initial value problem can be solved exactly with the help of the inverse scattering method, (II) they have an infinite number of conservation laws, (III) they have auto Bäcklund transformations, (IV) besides Lie point transformation groups they admit Lie Bäcklund transformation groups, (V) they describe pseudospherical surfaces, i.e. surfaces of constant negative gaussian curvature, (VI) they can be written as covariant exterior derivative of Lie algebra valued differential forms. It is conjectured that if the property (I) holds, then the properties (II) through (VI) also hold. We mention that there exist evolution equations which cannot be solved by the inverse scattering method, however these equations admit auto Bäcklund transformations. An example is the Liouville equation.

In the present paper we consider an evolution equation which admits Lie-Bäcklund transformation groups and its connection with the Painlevé property. We demonstrate for our particular example that the group theoretical reduction does not lead in general to an ordinary differential equation with the Painlevé property. Given a Lie point transformation group which is admitted by a given partial differential equation, there is a standard procedure for finding the solutions of the equation invariant under this group [8]. Here we use a different technique which can also be applied when we take into account Bäcklund transformation groups. As far as we know this technique is new. For our purpose we adopt the jet bundle formalism. The evolution equation under consideration is the non-linear diffusion equation [9]

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{u^2}\cdot\frac{\partial u}{\partial x}\right) \equiv \frac{1}{u^2}\cdot\frac{\partial^2 u}{\partial x^2} - \frac{2}{u^3}\left(\frac{\partial u}{\partial x}\right)^2 \equiv -\frac{\partial^2}{\partial x^2}\left(\frac{1}{u}\right). \quad (1)$$

Within the jet bundle formalism we consider instead of (1) the submanifold [10]

$$F(x, t, u, u_x, \ldots) \equiv u_t - u_{xx}/u^2 + 2u_x^2/u^3 = 0 \quad (2)$$

at its differential consequences

$$F_x(x, t, u, u_x, \ldots) \equiv u_{tx} - \frac{u_{xxx}}{u^2} + \frac{6u_x u_{xx}}{u^3} - \frac{6u_x^3}{u^4} = 0$$
$$\vdots \qquad \vdots \quad (3)$$

and the contact forms

$$\theta = du - u_x\,dx - u_t\,dt,$$
$$\theta_x = du_x - u_{xx}\,dx - u_{tx}\,dt \quad (4)$$
$$\vdots \qquad \vdots \quad .$$

The non-linear p.d.e. (1) admits the Lie point symmetries (infinitesimal generator)

$$X_v = -u_x\frac{\partial}{\partial u}, \qquad T_v = -u_t\frac{\partial}{\partial u},$$

Reprint requests to Dr. W.-H. Steeb, Theoretische Physik, Universität Paderborn, D-4790 Paderborn.

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$$S_v = (-x\,u_x - 2t\,u_t)\frac{\partial}{\partial u},$$
$$V_v = (x\,u_x + u)\frac{\partial}{\partial u}. \tag{5}$$

The subscript v denotes that we consider the vertical vector fields. The non-linear p.d.e. (1) also admits Lie-Bäcklund transformation groups. For example,

$$U = \left(\frac{u_{xxx}}{u^3} - \frac{9\,u_x\,u_{xx}}{u^4} + \frac{12\,u_x^3}{u^5}\right)\frac{\partial}{\partial u}. \tag{6}$$

For reducing (1) we consider a linear combination of the vector fields T_v and U, i.e. $a\,T_v + U$ ($a \in \mathbb{R}$). The equation

$$(a\,T_v + U) \lrcorner\, \theta = 0, \tag{7}$$

where $\lrcorner$ denotes the contraction, leads to the submanifold

$$-a\,u_t + \frac{u_{xxx}}{u^3} - \frac{9\,u_x\,u_{xx}}{u^4} + \frac{12\,u_x^3}{u^5} = 0. \tag{8}$$

Inserting (2) into (8) we find that

$$-a\left(\frac{u_{xx}}{u^2} - \frac{2\,u_x^2}{u^3}\right) + \frac{u_{xxx}}{u^3} - \frac{9\,u_x\,u_{xx}}{u^4} + \frac{12\,u_x^3}{u^5} = 0. \tag{9}$$

Consequently, it follows that

$$j\,s^*\left[-a\left(\frac{u_{xx}}{u^2} - \frac{2\,u_x^2}{u^3}\right) + \frac{u_{xxx}}{u^3} - \frac{9\,u_x\,u_{xx}}{u^4} + \frac{12\,u_x^3}{u^5}\right] \equiv \frac{\partial^3}{\partial x^3}\left(\frac{1}{2\,u^2}\right) - a\,\frac{\partial^2}{\partial x^2}\left(\frac{1}{u}\right) = 0, \tag{10}$$

where s is a cross section $s(x,t) = (x, t, u(x,t))$ with $j\,s^*\,\theta = 0$, $j\,s^*\,\theta_x = 0, \ldots$. $j\,s$ is the extension of s up to infinite order. For deriving (10) we have taken into account the identity

$$\frac{1}{u^3}\cdot\frac{\partial^3 u}{\partial x^3} - \frac{9}{u^4}\cdot\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial x^2} + \frac{12}{u^5}\left(\frac{\partial u}{\partial x}\right)^3 \equiv -\frac{\partial^3}{\partial x^3}\left(\frac{1}{2\,u^2}\right) \tag{11}$$

and the identity given by (1). Since derivatives of u with respect to t do not appear in (11) we are able to consider (11) as an ordinary differential equation of second order

$$\frac{d^3}{dx^3}\left(\frac{1}{2\,u^2}\right) - a\,\frac{d^2}{dx^2}\,\frac{1}{u} = 0, \tag{12}$$

where t plays the role of a parameter. The time t occurs in the constant of integration. The integration of (12) yields

$$\frac{du}{dx} + a\,u^2 = (C_1(t)\,x + C_2(t))\,u^3. \tag{13}$$

If one contrasts the solution of the equation $du/dx + u^2 = 0$, with that of the equation $du/dx + u^3 = 0$, one observes that the solution of the first equation has a movable pole, while that of the second has a movable branch point. Consequently, (13) does not have the Painlevé property. In order to determine the constants of integration C_1 and C_2 we must first solve the ordinary differential equation (13), where a new constant of integration appears which also depends on time. Then we insert the solution into the partial differential equation (1) and determine the quantities C_1, C_2, and C_3.

To sum up: Even if an evolution equation admits Lie-Bäcklund transformation groups we cannot conclude in general that the group theoretical reduction of the evolution equation leads to an ordinary differential equation with the Painlevé property. However, (1) can be directly linearized. Hence, one expects that any ordinary differential equation reduction of (1) will be of Painlevé type after a suitable transformation. This transformation can be found with the help of the singular point analysis.

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